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APPROXIMATELY BISECTRIX-ORTHOGONALITY PRESERVING MAPPINGS

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ABSTRACT. Regarding the geometry of a real normed space $\mathcal{X}$, we mainly introduce a notion of approximate bisectrix-orthogonality on vectors $x, y \in \mathcal{X}$ as follows:

$$x\,{}^{\varepsilon}\!\perp_W y \text{ if and only if } \sqrt{2}\frac{1-\varepsilon}{1+\varepsilon}\|x\|\,\|y\| \leq \Big\|\,\|y\|x + \|x\|y\,\Big\| \leq \sqrt{2}\frac{1+\varepsilon}{1-\varepsilon}\|x\|\,\|y\|.$$

We study class of linear mappings preserving the approximately bisectrix-orthogonality ${}^{\varepsilon}\!\perp_W$. In particular, we show that if $T : \mathcal{X} \to \mathcal{Y}$ is an approximate linear similarity, then

$$x\,{}^{\delta}\!\perp_W y \Longrightarrow Tx\,{}^{\theta}\!\perp_W Ty \qquad (x, y \in \mathcal{X})$$

for any $\delta \in [0, 1)$ and certain $\theta \geq 0$.

1. Introduction and preliminaries

There are several concepts of orthogonality appeared in the literature during the past century such as Birkhoff–James, Phythagorean, isosceles, Singer, Roberts, Diminnie, Carlsson, Rätz, ρ-orthogonality, area orthogonality, etc, in an arbitrary real normed space $\mathcal{X}$, which can be regarded as generalizations of orthogonality in the inner product spaces, in general [1, 2]. These are of intrinsic geometric interest and have been studied by many mathematicians. Among them we recall the following ones:

(i) *Birkhoff–James* $\perp_B$: $x \perp_B y$ if $\|x\| \leq \|x + ty\|$ for all scalars t (see [3]).
(ii) *Phythagorean* $\perp_P$: $x \perp_P y$ if $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ (see [9]).
(iii) *Isosceles* $\perp_I$: $x \perp_I y$ if $\|x + y\| = \|x - y\|$ (see [9, 14]).
(iv) *Roberts* $\perp_R$: $x \perp_R y$ if $\|x + ty\| = \|x - ty\|$ for all scalars t (see [15]).

The following mapping $\langle .|. \rangle_g : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ was introduced by Miličič [12]:

$$\langle y|x \rangle_g = \frac{1}{2}(\rho'_+(x, y) + \rho'_-(x, y)),$$

where mappings $\rho'_+, \rho'_- : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ are defined by

$$\rho'_\pm(x, y) = \lim_{t\to 0^\pm} \frac{\|x + ty\|^2 - \|x\|^2}{2t}.$$

In addition the ρ-orthogonality $x \perp_\rho y$ means $\langle y|x \rangle_g = 0$.
Note that $\perp_R, \perp_\rho \subseteq \perp_B$ [1] and the relations $\perp_P, \perp_I, \perp_B$ are, however, independent. If $(\mathcal{H}, \langle .|. \rangle)$ is a real inner product space, then all above relations coincide with usual orthogonality $\perp$ derived from $\langle .|. \rangle$ [2].
In the present note, we consider the so-called bisectrix-orthogonality and we study the orthogonality preserving property of this kind of orthogonality.

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Let $\mathcal{X}$ be a real normed space and $x, y \in \mathcal{X}$. The bisectrix-orthogonality relation $x \perp_W y$ (cf. Section 5.2 in [1]) is defined by

$$\Big\| \|y\|x + \|x\|y \Big\| = \sqrt{2}\|x\|\,\|y\|,$$

which for nonzero x and y means

$$\Big\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \Big\| = \sqrt{2}.$$

For instance consider the space $(\mathbb{R}^2, |||.|||)$ where $|||(r,s)||| = \max\{|r|,|s|\}$ for $(r,s) \in \mathbb{R}^2$. Then $(1,0) \perp_W (r,s)$ if and only if either (r,s) is the zero vector or

$$(r,s) \in (\{-\sqrt{2}-1, \sqrt{2}-1\}) \times [-\sqrt{2}, \sqrt{2}] \cup ([-\sqrt{2}-1, \sqrt{2}-1] \times \{-\sqrt{2}, \sqrt{2}\}).$$

Now we recall some properties of bisectrix-orthogonality (the proofs can be found in [1, Proposition 5.2.1]):

(P.1) $x \perp_W 0, 0 \perp_W y$ for all $x, y \in \mathcal{X}$;
(P.2) $x \perp_W y$ if and only if $y \perp_W x$;
(P.3) If $x \perp_W y$ and $x, y \neq 0$, then x, y are linearly independent;
(P.4) If $x \perp_W y$ and $\alpha\beta \geq 0$, then $\alpha x \perp_W \beta y$;
(P.5) In an inner product space, $x \perp_W y$ if and only if $\langle x|y\rangle = 0$.

By the definition of bisectrix-orthogonality and Pythagorean orthogonality one can easily get the following properties:

(P.6) For all nonzero vectors $x, y \in \mathcal{X}$, $x \perp_W y$ if and only if $\frac{x}{\|x\|} \perp_P \frac{y}{\|y\|}$;
(P.7) For all $x, y \in S_{\mathcal{X}} = \{z \in \mathcal{X} : \|z\| = 1\}, x \perp_W y$ if and only if $x \perp_P y$.

We state some relations between bisectrix-orthogonality and other orthogonalities. It is known [1, 2] that each of the following properties implies that the norm $\|.\|$ comes from an inner product.

(P.8) $\perp_W \subseteq \perp_I$ over $\mathcal{X}$;
(P.9) $\perp_I \subseteq \perp_W$ over $\mathcal{X}$;
(P.10) $\perp_W \subseteq \perp_B$ over $S_{\mathcal{X}}$;
(P.11) $\perp_B \subseteq \perp_W$ over $S_{\mathcal{X}}$;
(P.12) If for all $x, y \in \mathcal{X}, x \perp_W y$ implies $\|x+y\|^2 + \|x-y\|^2 \sim 2\|x\|^2 + 2\|y\|^2$

where $\sim$ stands either for $\leq$ or $\geq$. An easy consequence of $\perp_R, \perp_\rho \subseteq \perp_B$ states that if for all $x, y \in S_{\mathcal{X}}$, the relation $x \perp_W y$ implies $x \perp_R y$ or $x \perp_\rho y$, then the norm $\|.\|$ comes from an inner product.

In this paper we introduce two notions of approximate bisectrix-orthogonality ${}^{\varepsilon}\!\perp_W$ and $\perp_W^{\varepsilon}$ in a real normed space $\mathcal{X}$ and study the class of linear mappings which preserve the approximately bisectrix-orthogonality of type ${}^{\varepsilon}\!\perp_W$. In particular, we show that if $T : \mathcal{X} \to \mathcal{Y}$ is an approximate linear similarity, then

$$x\,{}^{\delta}\!\perp_W y \Longrightarrow Tx\,{}^{\theta}\!\perp_W Ty \qquad (x, y \in \mathcal{X})$$

for any $\delta \in [0,1)$ and certain $\theta \geq 0$.

2. Approximately bisectrix-orthogonality preserving mappings

Let ζ, η be elements of an inner product space $(\mathcal{H}, \langle .|.\rangle)$ and $\varepsilon \in [0,1)$. The approximate orthogonality $\zeta \perp^{\varepsilon} \eta$ defined by

$$|\langle \zeta|\eta\rangle| \leq \varepsilon\|\zeta\|\,\|\eta\|$$

or equivalently,

$$|\cos(\zeta, \eta)| \leq \varepsilon \quad (\zeta, \eta \neq 0).$$

So, it is natural to generalize the notion of approximate orthogonality for a real normed space $\mathcal{X}$. This fact motivated Chmieliński and Wójcik [5] to give for two elements $x, y \in \mathcal{X}$ the following definition of the approximate isosceles-orthogonality $x\,{}^{\varepsilon}\!\perp_I y$ as follows:

$$\Big|\, \|x+y\| - \|x-y\| \,\Big| \leq \varepsilon(\|x+y\| + \|x-y\|).$$

They also introduced another approximate isosceles-orthogonality $x \perp_I^{\varepsilon} y$ by

$$\Big|\, \|x+y\|^2 - \|x-y\|^2 \,\Big| \leq 4\varepsilon\|x\|\|y\|.$$

Inspired by the above approximate isosceles-orthogonality, we propose two definitions of approximate bisectrix-orthogonality.
Let $\varepsilon \in [0,1)$ and $x, y \in \mathcal{X}$, let us put $x\,{}^{\varepsilon}\!\perp_W y$ if

$$\Big|\Big\| \|y\|x + \|x\|y \Big\| - \sqrt{2}\|x\|\,\|y\|\Big| \leq \varepsilon\Big(\Big\| \|y\|x + \|x\|y \Big\| + \sqrt{2}\|x\|\,\|y\|\Big)$$

or equivalently,

$$\sqrt{2}\frac{1-\varepsilon}{1+\varepsilon}\|x\|\,\|y\| \leq \Big\| \|y\|x + \|x\|y \Big\| \leq \sqrt{2}\frac{1+\varepsilon}{1-\varepsilon}\|x\|\,\|y\|,$$

which means

$$\sqrt{2}\frac{1-\varepsilon}{1+\varepsilon} \leq \Big\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\Big\| \leq \sqrt{2}\frac{1+\varepsilon}{1-\varepsilon}$$

for nonzero vectors x and y.
Also we define $x \perp_W^{\varepsilon} y$ if

$$\Big|\Big\| \|y\|x + \|x\|y \Big\|^2 - 2\|x\|^2\,\|y\|^2 \Big| \leq 2\varepsilon\|x\|^2\,\|y\|^2$$

or equivalently,

$$\sqrt{2(1-\varepsilon)}\|x\|\,\|y\| \leq \Big\| \|y\|x + \|x\|y \Big\| \leq \sqrt{2(1+\varepsilon)}\|x\|\,\|y\|,$$

which means

$$\sqrt{2(1-\varepsilon)} \leq \Big\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \Big\| \leq \sqrt{2(1+\varepsilon)}$$

for nonzero $x, y \in \mathcal{X}$.
It is easy to check that in the case where the norm comes from a real-valued inner product we have

$$x\,{}^{\varepsilon}\!\perp_W y \Leftrightarrow |\langle x, y\rangle| \leq \frac{4\varepsilon}{(1-\varepsilon)^2}\|x\|\,\|y\|$$

and

$$x \perp_W^{\varepsilon} y \Leftrightarrow |\langle x, y\rangle| \leq \varepsilon\|x\|\,\|y\| \Leftrightarrow x \perp^{\varepsilon} y$$

Thus the second approximate bisectrix-orthogonality coincides with the natural notion of approximate orthogonality for inner product spaces.
Note that the relations ${}^{\varepsilon}\!\perp_W$ and $\perp_W^{\varepsilon}$ are symmetric and almost homogeneous in the sense that

$$x\,{}^{\varepsilon}\!\perp_W y \Longrightarrow y\,{}^{\varepsilon}\!\perp_W x \;\text{ and }\; \alpha x\,{}^{\varepsilon}\!\perp_W \beta y \;\text{ for }\; \alpha\beta \geq 0$$

and

$$x \perp_W^{\varepsilon} y \Longrightarrow y \perp_W^{\varepsilon} x \;\text{ and }\; \alpha x \perp_W^{\varepsilon} \beta y \;\text{ for }\; \alpha\beta \geq 0.$$

Remark 2.1. It is easy to see that $\perp_W^\varepsilon$ implies ${}^\varepsilon\!\perp_W$ with the same ε. Also if $\varepsilon \in [0, \frac{1}{16})$, then ${}^\varepsilon\!\perp_W$ implies $\perp_W^{16\varepsilon}$. Indeed, for $x, y \neq 0$, since $0 \leq \varepsilon < \frac{1}{16}$, so $\frac{1+\varepsilon}{1-\varepsilon} \leq \sqrt{1+16\varepsilon}$ and $\sqrt{1-16\varepsilon} \leq \frac{1-\varepsilon}{1+\varepsilon}$. Hence $x \,{}^\varepsilon\!\perp_W y$ implies

$$\sqrt{2(1-16\varepsilon)} \leq \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \leq \sqrt{2(1+16\varepsilon)}$$

or equivalently, $x \perp_W^{16\varepsilon} y$.

Now, suppose that $\mathcal{X}$ and $\mathcal{Y}$ are real normed spaces of dimensions greater than or equal to two and let $\delta, \varepsilon \in [0, 1)$. We say that a linear mapping $T : \mathcal{X} \to \mathcal{Y}$ preserves the approximate bisectrix-orthogonality if

$$x \,{}^\delta\!\perp_W y \Longrightarrow Tx \,{}^\varepsilon\!\perp_W Ty \qquad (x, y \in \mathcal{X}).$$

Notice that if $\delta = \varepsilon = 0$, we have

$$x \perp_W y \Longrightarrow Tx \perp_W Ty \qquad (x, y \in \mathcal{X}),$$

and we say that T preserves the bisectrix-orthogonality.
Koldobsky [10] (for real spaces) and Blanco and Turnšek [4] (for real and complex ones) proved that a linear mapping $T : \mathcal{X} \to \mathcal{Y}$ preserving the Birkhoff orthogonality has to be a similarity, i.e., a non-zero-scalar multiple of an isometry. Further, Chmieliński and Wójcik [7, 16] proved that a linear mapping $T : \mathcal{X} \to \mathcal{Y}$ preserving the ρ-orthogonality has to be a similarity. Approximately orthogonality preserving mappings in the framework of normed spaces have been recently studied. In the case where $\delta = 0$, Mojškerc and Turnšek [13] and Chmieliński [6] verified the properties of mappings that preserve approximate Birkhoff orthogonality. Also Chmieliński and Wójcik [5, 7] studied some properties of mappings that preserve approximate isosceles-orthogonality and ρ-orthogonality in the case when $\delta = 0$. Recently Zamani and Moslehian [17] studied approximate Roberts orthogonality preserving mappings.
The next lemma plays an essential role in our work. It provides indeed a reverse of the triangle inequality; see [8].

Lemma 2.2. [11, Theorem 1] *Let $\mathcal{X}$ be a normed space and $x, y \in \mathcal{X} \setminus \{0\}$. Then*

$$\begin{aligned}\|x\| + \|y\| + \Big(\Big\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \Big\| - 2\Big) \max\{\|x\|, \|y\|\} \\ &\leq \|x+y\| \\ &\leq \|x\| + \|y\| + \Big(\Big\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \Big\| - 2\Big) \min\{\|x\|, \|y\|\}\end{aligned}$$

To reach our main result, we need some lemmas, which are interesting on their own right. We state some prerequisites for the first lemma. For a bounded linear mapping $T : \mathcal{X} \to \mathcal{Y}$, let $\|T\| = \sup\{\|Tx\|;\ \|x\| = 1\}$ denote the operator norm and $[T] := \inf\{\|Tx\|;\ \|x\| = 1\}$. Notice that for any $x \in \mathcal{X}$, we have $[T]\|x\| \leq \|Tx\| \leq \|T\|\|x\|$.

Lemma 2.3. *Let $\delta, \varepsilon \in [0, 1)$. If a nonzero bounded linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\frac{1-\varepsilon}{1+\varepsilon}\gamma\|x\| \leq \|Tx\| \leq \frac{1+\varepsilon}{1-\varepsilon}\gamma\|x\|$$

for all $x \in \mathcal{X}$ and all $\gamma \in \left[\frac{1-\delta}{1+\delta}[T], \frac{1+\delta}{1-\delta}\|T\|\right]$, then

$$x \,{}^\delta\!\perp_W y \Longrightarrow Tx \,{}^\varepsilon\!\perp_W Ty \qquad (x, y \in \mathcal{X}).$$

Proof. Let $x, y \in \mathcal{X} \setminus \{0\}$ and $x \,{}^{\delta}\!\!\perp_W y$. Then $\sqrt{2}\frac{1-\delta}{1+\delta} \leq \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| \leq \sqrt{2}\frac{1+\delta}{1-\delta}$. If $x = sy$ for some $s \in \mathbb{R} \setminus \{0\}$, then for $\gamma = \frac{1-\delta}{1+\delta}[T]$ we have

$$\begin{aligned}
\left\|\frac{Tx}{\|Tx\|} + \frac{Ty}{\|Ty\|}\right\| &= \left\|T\Big(\frac{x}{\|Tx\|} + \frac{y}{\|Ty\|}\Big)\right\| \\
&= \left\|T\Big(\frac{sy}{\|sTy\|} + \frac{y}{\|Ty\|}\Big)\right\| \\
&= \frac{\|y\|}{\|Ty\|}\left\|T\Big(\frac{sy}{\|sy\|} + \frac{y}{\|y\|}\Big)\right\| \\
&= \frac{\|y\|}{\|Ty\|}\left\|T\Big(\frac{x}{\|x\|} + \frac{y}{\|y\|}\Big)\right\| \\
&\leq \frac{\|y\|}{\|Ty\|}\frac{1+\varepsilon}{1-\varepsilon}\gamma\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| \\
&= \frac{\|y\|}{\|Ty\|}\frac{1+\varepsilon}{1-\varepsilon}\frac{1-\delta}{1+\delta}[T]\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| \\
&\leq \frac{1+\varepsilon}{1-\varepsilon}\frac{1-\delta}{1+\delta}\sqrt{2}\frac{1+\delta}{1-\delta} \\
&= \sqrt{2}\frac{1+\varepsilon}{1-\varepsilon},
\end{aligned}$$

whence $\left\|\frac{Tx}{\|Tx\|} + \frac{Ty}{\|Ty\|}\right\| \leq \sqrt{2}\frac{1+\varepsilon}{1-\varepsilon}$. Similarly, $\sqrt{2}\frac{1-\varepsilon}{1+\varepsilon} \leq \left\|\frac{Tx}{\|Tx\|} + \frac{Ty}{\|Ty\|}\right\|$. Thus $Tx \,{}^{\varepsilon}\!\!\perp_W Ty$.
Assume that x, y are linearly independent. Set $\gamma_0 := \frac{\sqrt{2}}{\left\|\frac{x}{\|Tx\|} + \frac{y}{\|Ty\|}\right\|}$. We may assume that $\frac{\|x\|}{\|Tx\|} \leq \frac{\|y\|}{\|Ty\|}$. By Lemma 2.2 we have

$$\begin{aligned}
\left\|\frac{x}{\|Tx\|} + \frac{y}{\|Ty\|}\right\| &\leq \frac{\|x\|}{\|Tx\|} + \frac{\|y\|}{\|Ty\|} + \Big(\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| - 2\Big)\min\{\frac{\|x\|}{\|Tx\|}, \frac{\|y\|}{\|Ty\|}\} \\
&\leq \frac{\|y\|}{\|Ty\|} + \Big(\sqrt{2}\frac{1+\delta}{1-\delta} - 1\Big)\frac{\|x\|}{\|Tx\|} \\
&\leq \frac{1}{[T]} + \Big(\sqrt{2}\frac{1+\delta}{1-\delta} - 1\Big)\frac{1}{[T]} \\
&= \sqrt{2}\frac{1+\delta}{1-\delta}\frac{1}{[T]}.
\end{aligned}$$

So that $\gamma_0 \geq \frac{\sqrt{2}}{\sqrt{2}\frac{1+\delta}{1-\delta}\frac{1}{[T]}} = \frac{1-\delta}{1+\delta}[T]$.
Similarly we get $\gamma_0 \leq \frac{1+\delta}{1-\delta}\|T\|$. Thus $\gamma_0 \in \left[\frac{1-\delta}{1+\delta}[T], \frac{1+\delta}{1-\delta}\|T\|\right]$. Our hypothesis implies that

$$\frac{1-\varepsilon}{1+\varepsilon}\gamma_0\|z\| \leq \|Tz\| \leq \frac{1+\varepsilon}{1-\varepsilon}\gamma_0\|z\| \qquad (z \in \mathcal{X})$$

or equivalently,

$$\Big|\|Tz\| - \gamma_0\|z\|\Big| \leq \varepsilon(\|Tz\| + \gamma_0\|z\|) \qquad (z \in \mathcal{X})$$

Putting $\|Ty\|x+\|Tx\|y$ instead of z in the above inequality we get

$$\begin{aligned}\Big|\Big\|\,\|Ty\|Tx+\|Tx\|Ty\,\Big\| - \frac{\sqrt{2}}{\Big\|\frac{x}{\|Tx\|}+\frac{y}{\|Ty\|}\Big\|}\Big\|\,\|Ty\|x+\|Tx\|y\,\Big\|\Big| \\ \leq \varepsilon\Big(\Big\|\,\|Ty\|Tx+\|Tx\|Ty\,\Big\| + \frac{\sqrt{2}}{\Big\|\frac{x}{\|Tx\|}+\frac{y}{\|Ty\|}\Big\|}\Big\|\,\|Ty\|x+\|Tx\|y\,\Big\|\Big).\end{aligned}$$

Thus

$$\Big|\Big\|\,\|Ty\|Tx+\|Tx\|Ty\,\Big\| - \sqrt{2}\|Tx\|\,\|Ty\|\,\Big| \leq \varepsilon\Big(\Big\|\,\|Ty\|Tx+\|Tx\|Ty\,\Big\| + \sqrt{2}\|Tx\|\,\|Ty\|\Big),$$

whence $Tx \,{}^{\varepsilon}\!\!\perp_W Ty$. ☐

Lemma 2.4. *Let $\delta, \varepsilon \in [0,1)$. If a nonzero bounded linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfies $\frac{1+\delta}{1-\delta}\|Tz\|\,\|u\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|Tu\|\,\|z\|$ for all $z, u \in \mathcal{X}$, then*

$$x \,{}^{\delta}\!\!\perp_W y \Longrightarrow Tx \,{}^{\varepsilon}\!\!\perp_W Ty \qquad (x, y \in \mathcal{X}).$$

Proof. By our assumption we have, $\frac{1+\delta}{1-\delta}\|Tz\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|Tu\|$ for all z, u with $\|z\| = \|u\| = 1$. Passing to the infimum over $\|u\| = 1$, we get

$$\frac{1+\delta}{1-\delta}\|Tz\| \leq \frac{1+\varepsilon}{1-\varepsilon}[T] \quad (\|z\| = 1),$$

and passing to the supremum over $\|z\| = 1$ we obtain

$$\frac{1+\delta}{1-\delta}\|T\| \leq \frac{1+\varepsilon}{1-\varepsilon}[T].$$

Now, let $\gamma \in \Big[\frac{1-\delta}{1+\delta}[T], \frac{1+\delta}{1-\delta}\|T\|\Big]$ and $x \in \mathcal{X}$. Therefore we have

$$\begin{aligned}\frac{1-\varepsilon}{1+\varepsilon}\gamma\|x\| &\leq \frac{1-\varepsilon}{1+\varepsilon} \times \frac{1+\delta}{1-\delta}\|T\|\,\|x\| \\ &\leq \frac{1-\varepsilon}{1+\varepsilon} \times \frac{1+\delta}{1-\delta} \times \frac{1+\varepsilon}{1-\varepsilon} \times \frac{1-\delta}{1+\delta}[T]\,\|x\| \\ &\leq \|Tx\| \\ &\leq \|T\|\,\|x\| \\ &\leq \frac{1+\varepsilon}{1-\varepsilon} \times \frac{1-\delta}{1+\delta}[T]\,\|x\| \\ &\leq \frac{1+\varepsilon}{1-\varepsilon} \times \frac{1-\delta}{1+\delta} \times \frac{1+\delta}{1-\delta}\gamma\|x\| \\ &= \frac{1+\varepsilon}{1-\varepsilon}\gamma\|x\|.\end{aligned}$$

Thus

$$\frac{1-\varepsilon}{1+\varepsilon}\gamma\|x\| \leq \|Tx\| \leq \frac{1+\varepsilon}{1-\varepsilon}\gamma\|x\|$$

Making a use of Lemma 2.3 just completes the proof. ☐

We are now in position to establish the main result. Following [13], we say that a linear mapping $U : \mathcal{X} \to \mathcal{Y}$ is an approximate linear isometry if

$$(1-\varphi_1(\varepsilon))\|z\| \leq \|Uz\| \leq (1+\varphi_2(\varepsilon))\|z\| \quad (z \in \mathcal{X}),$$

where $\varphi_1(\varepsilon) \to 0$ and $\varphi_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Notice that if $\varphi_1(\varepsilon) = \varphi_2(\varepsilon) = 0$, then U is an isometry.

A linear mapping $U : \mathcal{X} \to \mathcal{Y}$ is said to be an approximate similarity if it is a non-zero-scalar multiple of an approximate linear isometry, or equivalently it satisfies

$$\lambda(1-\varphi_1(\varepsilon))\|w\| \le \|Uw\| \le \lambda(1+\varphi_2(\varepsilon))\|w\|$$

for some unitary U, some $\lambda > 0$ and for all $w \in \mathcal{X}$, where $\varphi_1(\varepsilon) \to 0$ and $\varphi_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Theorem 2.5. *Let $U : \mathcal{X} \to \mathcal{Y}$ be an approximate linear similarity and $\delta \in [0,1)$. If a nonzero bounded linear mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfies $\|T - U\| \le \varepsilon\|U\|$, then*

$$x \,{}^{\delta}\!\perp_W y \Longrightarrow Tx \,{}^{\theta}\!\perp_W Ty \qquad (x, y \in \mathcal{X}),$$

where $\theta = \frac{2\delta+2\varepsilon+(1-\delta)\varphi_1(\varepsilon)+(1+\delta+2\varepsilon)\varphi_2(\varepsilon)}{2+2\delta\varepsilon-(1-\delta)\varphi_1(\varepsilon)+(1+\delta+2\delta\varepsilon)\varphi_2(\varepsilon)}$.

Proof. For any $w \in \mathcal{X}$ we have

$$\Big|\, \|Tw\| - \|Uw\| \,\Big| \le \|Tw - Uw\| \le \|T - U\|\,\|w\| \le \varepsilon\|U\|\,\|w\| \le \varepsilon\lambda(1+\varphi_2(\varepsilon))\|w\|,$$

whence

$$-\varepsilon\lambda(1+\varphi_2(\varepsilon))\|w\| \le \|Tw\| - \|Uw\| \le \varepsilon\lambda(1+\varphi_2(\varepsilon))\|w\|.$$

Since

$$\lambda(1-\varphi_1(\varepsilon))\|w\| \le \|Uw\| \le \lambda(1+\varphi_2(\varepsilon))\|w\|,$$

therefore we get

$$\lambda\Big[(1-\varphi_1(\varepsilon)) - \varepsilon(1+\varphi_2(\varepsilon))\Big]\|w\| \le \|Tw\| \le \lambda(1+\varepsilon)(1+\varphi_2(\varepsilon))\|w\|.$$

Thus for any $z, u \in \mathcal{X}$, we have

$$\begin{aligned}
\frac{1+\delta}{1-\delta}\|Tz\|\,\|u\| &\le \frac{1+\delta}{1-\delta}\lambda(1+\varepsilon)(1+\varphi_2(\varepsilon))\|z\|\,\frac{\|Tu\|}{\lambda\Big[(1-\varphi_1(\varepsilon)) - \varepsilon(1+\varphi_2(\varepsilon))\Big]}\\
&= \frac{(1+\varepsilon)(1+\varphi_2(\varepsilon))(1+\delta)}{[(1-\varphi_1(\varepsilon)) - \varepsilon(1+\varphi_2(\varepsilon))](1-\delta)}\|Tu\|\|z\|\\
&= \frac{1+\frac{2\delta+2\varepsilon+(1-\delta)\varphi_1(\varepsilon)+(1+\delta+2\varepsilon)\varphi_2(\varepsilon)}{2+2\delta\varepsilon-(1-\delta)\varphi_1(\varepsilon)+(1+\delta+2\delta\varepsilon)\varphi_2(\varepsilon)}}{1-\frac{2\delta+2\varepsilon+(1-\delta)\varphi_1(\varepsilon)+(1+\delta+2\varepsilon)\varphi_2(\varepsilon)}{2+2\delta\varepsilon-(1-\delta)\varphi_1(\varepsilon)+(1+\delta+2\delta\varepsilon)\varphi_2(\varepsilon)}}\|Tu\|\|z\|\\
&= \frac{1+\theta}{1-\theta}\|Tu\|\|z\|.
\end{aligned}$$

Therefore $\frac{1+\delta}{1-\delta}\|Tz\|\,\|u\| \le \frac{1+\theta}{1-\theta}\|Tu\|\,\|z\|$. Now the assertion follows from Lemma 2.4. $\square$

As a consequence, with $\varepsilon = 0$ and $T = U$, we have

Corollary 2.6. *Let $T : \mathcal{X} \to \mathcal{Y}$ be an approximate linear similarity. Then*

$$x \,{}^{\delta}\!\perp_W y \Longrightarrow Tx \,{}^{\theta}\!\perp_W Ty \qquad (x, y \in \mathcal{X})$$

for any $\delta \in [0,1)$, where $\theta = \frac{2\delta+(1-\delta)\varphi_1(\varepsilon)+(1+\delta)\varphi_2(\varepsilon)}{2-(1-\delta)\varphi_1(\varepsilon)+(1+\delta)\varphi_2(\varepsilon)}$.

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